# Route to chaotic synchronisation in coupled map lattices: Rigorous results

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**Abstract**

Two-dimensional mappings obtained by coupling two piecewise increasing expanding maps are considered. Their dynamics is described when the coupling parameter increases in the expanding domain. By introducing a coding and by analysing an admissibility condition, approximations of the corresponding symbolic systems are obtained. These approximations imply that the topological entropy is located between two decreasing step functions of the coupling parameter. The analysis firstly applies to mappings with piecewise affine local maps which allow explicit expressions and, in a second step, is extended by continuity to mappings with piecewise smooth local maps.

## 1 Introduction

Coupled map lattices (CML) are discrete models for the nonlinear dynamics of some extended systems. For instance, they have been used to model population dynamics, reaction-diffusion systems or alloy solidification processes. We refer to the article by Kaneko in [11] for a review of applications of CML. As dynamical systems with multidimensional (infinite dimensional) phase space, CML are also interesting in their own because they often do no satisfy usual assumptions such as uniform hyperbolicity or existence of Markov partition.

The dynamics of CML is generated by a mapping with two components. One component is a mapping of the real variable, the local map, which represents a local forcing at each site of the lattice. The other component is a convex combination of local states, the coupling operator, which represents a diffusive interaction between sites. A parameter, called coupling, measures the strength of the coupling operator. (The coupling is equal to zero means that the coupling operator is the identity, i.e. that the system is uncoupled.) One of the main question in the theory of CML is to describe the dynamics when the coupling increases from zero [3].

For CML with expanding local map, in the domain of couplings where the mapping keeps expanding, the dynamics has mainly been mathematically described for (very) small couplings. In such a domain, the complete chaotic structure of the uncoupled system persists. That is to say, there exists an absolutely continuous invariant measure with decay of correlations [5, 10] and/or the system is topologically conjugated to the uncoupled system [1]. This SRB-measure needs not be unique if the lattice has an infinite number of sites [9].

For stronger coupling but still when the mapping is expanding, effects of diffusive interaction modify the dynamics. From the ergodic point of view, phase transitions are expected [4] and were proved to occur in CML with special coupling operator [8]. From the topological point of view, the complexity of the system (e.g. its topological entropy) might decrease with coupling but this decay has not been mathematically proved. One reason is that excepted in special examples [7] where a Markov partition and the subsequent transition matrix have been determined, the symbolic dynamics associated to CML remained unknown.

When the coupling increases further so that the mapping ceases to be expanding, chaotic synchronisation takes place [14]. It means that the diagonal of the phase space attracts (at least) a set of initial conditions of positive Lebesgue measure [2, 13].

These results confirm that the dynamics fundamentally differs between the small coupling domain and the synchronisation region. However, changes in the dynamics when the coupling increases between these two domains, the route to synchronisation for short, is not so well known. The present paper is devoted to the description of these changes in some CML with two sites. The local maps are chosen expanding, piecewise increasing and similar to Lorenz maps [15]. The route to synchronisation is described in the framework of symbolic dynamics and translated in terms of topological entropy. By using a natural coding, upper and lower estimates of the set of symbolic sequences which imply that the topological entropy is located between two decreasing step functions of the coupling. It means that the complexity of the coupled map lattice becomes smaller and approaches the complexity of the synchronised system when the coupling increases.

The plan of the paper is as follows. We firstly consider piecewise affine maps. In Section 2, we obtain an explicit expression of the admissibility condition of symbolic sequences and we prove that the symbolic system is conjugated to the CML on its repeller (Proposition 2.1). Moreover, we establish a partial ordering on symbolic sequences compatible with the ordering of the coordinates of their images (Proposition 2.2). With these tools provided, we obtain an upper bound and a lower bound for the set of admissible sequences which both decrease with coupling (Theorem 3.2). Computing the corresponding topological entropies gives upper and lower bounds for entropy of the CML (Figure 3). In addition monotonicity of the set of admissible sequences in a neighbourhood of the uncoupled domain is also shown. All these results are given Section 3 and their proofs are given in Sections 4 and 5. In a second step (Section 6), by proving a kind of $C^1$-structural stability for piecewise increasing expanding CML, the previous bounds are extended to CML with smooth local maps. Precisely, Theorem 6.2 shows that small perturbations of the local map implies small perturbations of the complete dynamical picture in the coupling domain.

## 2 Piecewise affine expanding CML and their coding

Coupled map lattices of two sites are discrete dynamical systems in $\mathbb{R}^2$ generated by the one-parameter family of maps $F_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F_\epsilon(x_0, x_1) = ((1-\epsilon)f(x_0) + \epsilon f(x_1), (1-\epsilon)f(x_1) + \epsilon f(x_0)), \tag{1}$$

for all $(x_0, x_1) \in \mathbb{R}^2$. In this paper, we consider local maps defined by $f(x) = ax + (1-a)H(x-1/2)$ for all $x \in \mathbb{R}$ where $a > 2$ and where $H$ is the Heaviside function, i.e.

$$H(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geqslant 0 \end{cases}$$

see Figure 1. The coupling parameter $\epsilon \in [0, 1/2]$.

The dynamical system in $\mathbb{R}$ generated by the map $f$ has the following properties. The set $K = \bigcap\limits_{t\in\mathbb{N}} f^{-t}([0,1])$ is an invariant Cantor set which contains all points with bounded orbit under $f$. The set $K$ is called the repeller of the dynamical system $(\mathbb{R}, f)$. Moreover, the dynamical system $(K, f)$ is topologically conjugated to the full shift on symbolic sequences of 0's and 1's. The coding map associates to each point $x \in K$ a sequence of 0's and 1's using the position of the iterates $f^t(x)$ with respect to 1/2. (Conjugated means the existence of a bijection between two systems and topologically conjugated means that this bijection is a homeomorphism.)

These results can be obtained using nested pre-images of $[0,1]$ by $f$ to construct each point of $K$. But since the map is piecewise affine, one can use another method. The method consists in solving the recurrence induced by $f$ to obtain a formal expression of points in $\mathbb{R}$ which depends on symbolic sequences. This expression is employed to deduce a condition for a symbolic sequence to code a point in $K$ and to prove conjugacy to the symbolic system. Finally, the analysis of these condition and expression show that the symbolic system is the full shift and that $K$ is a Cantor set. (This method and the results are contained in the analysis of the CML below when considering orbits lying in the diagonal $\{(x,x) \ : \ x \in \mathbb{R}\}$. Indeed choosing $x_0 = x_1$ in relation (1) shows that the dynamics in the diagonal is given by $f$.)

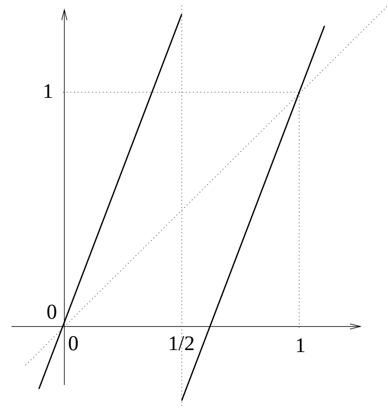


Figure 1: The local map $f$.

Because of the definition of $F_\epsilon$, one can apply this second method to the coupled map lattice to obtain a symbolic description of its repeller. In the next section, we present the first steps which consist in obtaining the expression of points from their code, the admissibility condition and in proving the conjugacy between the dynamics in the repeller and the symbolic system.

## 2.1 Symbolic description of the coupled map lattice

To express points in the repeller of the coupled map lattice using symbolic sequences, we need to introduce some notations. Given a point $(x_0, x_1) \in \mathbb{R}^2$, let $\{\mathbf{x}^t\}_{t\in\mathbb{N}}$ be the orbit issued from this point. It means that $\mathbf{x}^0 = (x_0, x_1)$, $\mathbf{x}^t = (x_0^t, x_1^t)$ and $\mathbf{x}^{t+1} = F_\epsilon(\mathbf{x}^t)$ for all $t \in \mathbb{N} = \{0, 1, 2, \cdots\}$. Let also $\theta = \{\theta^t\}_{t\in\mathbb{N}} = \{\theta_0^t\theta_1^t\}_{t\in\mathbb{N}} \in \Omega$, where $\Omega = \{00, 01, 10, 11\}^{\mathbb{N}}$, be the corresponding code given by

$$\theta_s^t = H(x_s^t - 1/2), \tag{2}$$

for all $t \in \mathbb{N}$ and $s \in \{0, 1\}$. (We have chosen the notation $\theta_0^t\theta_1^t$ rather than $(\theta_0^t, \theta_1^t)$ for the sake of simplicity in the sequel.) The set $\Omega$ is endowed with the product topology of the discrete topology on $\{00, 01, 10, 11\}$.

Consider the norm $\|(x_0, x_1)\| = \max\{|x_0|, |x_1|\}$ in $\mathbb{R}^2$. The **repeller** of the CML is the set $\mathcal{K}_\epsilon$ of points $(x_0, x_1) \in \mathbb{R}^2$ with bounded orbit, i.e. for which $\sup\limits_{t\in\mathbb{N}} \|\mathbf{x}^t\| < +\infty$.

Let now $\epsilon_a = (a-1)/(2a) > 1/4$ and, for every $\epsilon \in [0, \epsilon_a)$, consider the map $\psi_\epsilon : \Omega \to \mathbb{R}$ defined by

$$\psi_\epsilon(\theta) = (a-1)\sum_{k=0}^{+\infty} a^{-(k+1)}\left(\ell_0^{(k)}\theta_0^k + \ell_1^{(k)}\theta_1^k\right),$$

where $\ell_0^{(k)} = (1 + (1-2\epsilon)^{-k})/2$ and $\ell_1^{(k)} = (1 - (1-2\epsilon)^{-k})/2$ for all $k \in \mathbb{N}$. The condition $\epsilon \in [0, \epsilon_a)$ ensures that this series converges for every $\theta \in \Omega$. (A computation of the derivative $DF_\epsilon$ of $F_\epsilon$ shows that the condition $\epsilon \in [0, \epsilon_a)$ is equivalent to the condition that both eigenvalues of $DF_\epsilon$ have modulus larger than 1 and hence to the condition that the mapping $F_\epsilon$ is expanding.) **In the rest of the paper, we assume that $\epsilon \in [0, \epsilon_a)$.**

Finally, we introduce the shift $\sigma$ and the spatial flip $R$ acting in $\Omega$. Given a symbolic sequence $\theta$, let $(\sigma\theta)^t = \theta^{t+1}$ and $(R\theta)^t = \theta_1^t\theta_0^t$ for every $t \in \mathbb{N}$. Note that $\sigma R = R\sigma$.

The map $\psi_\epsilon$ is used to obtain the expression of points in $\mathcal{K}_\epsilon$ and a condition for a symbolic sequence to code for a point in this set. Given $\epsilon \in [0, \epsilon_a)$, let $\mathcal{A}_\epsilon \subset \Omega$ be the set of sequences $\theta$ which satisfy the following **admissibility condition**

$$\theta_s^t = H(\psi_\epsilon(\sigma^t R^s\theta) - 1/2), \tag{3}$$

for all $t \in \mathbb{N}$ and $s \in \{0, 1\}$.

With all these notations provided, we can state the conjugacy between the symbolic systems and the CML restricted to its repeller. Given $\theta \in \Omega$, let $\phi_\epsilon(\theta) = (\psi_\epsilon(\theta), \psi_\epsilon(R\theta))$.

**Proposition 2.1** *The map $\phi_\epsilon$ is a uniformly continuous bijection from $\mathcal{A}_\epsilon$ to $\mathcal{K}_\epsilon$ and the following relation holds*

$$F_\epsilon \circ \phi_\epsilon\big|_{\mathcal{A}_\epsilon} = \phi_\epsilon \circ \sigma\big|_{\mathcal{A}_\epsilon}.$$

Despite this result, the topological entropies of $(\mathcal{A}_\epsilon, \sigma)$ and $(\mathcal{K}_\epsilon, F_\epsilon)$ may differ if $\mathcal{A}_\epsilon$ is not compact and this can happen as condition (3) suggests. (For a recent discussion and references on the topological entropy on non-compact sets, we refer to [17].)

*Proof of the proposition:* The fact that $\phi_\epsilon$ is uniformly continuous is an immediate consequence of uniform convergence in the series defining $\psi_\epsilon$.

We now prove that, for every point $(x_0, x_1) \in \mathcal{K}_\epsilon$, there exists a sequence $\theta \in \mathcal{A}_\epsilon$ such that $(x_0, x_1) = \phi_\epsilon(\theta)$.

Given $(x_0, x_1) \in \mathbb{R}^2$, let $L_\epsilon(x_0, x_1) = ((1-\epsilon)x_0 + \epsilon x_1, (1-\epsilon)x_1 + \epsilon x_0)$. This operator $L_\epsilon$ is invertible when $\epsilon < 1/2$ and $\|L_\epsilon^{-1}\| = (1-2\epsilon)^{-1}$.

Let $(x_0, x_1) \in \mathcal{K}_\epsilon$ and let $\theta$ be the symbolic sequence obtained by applying relation (2). The definition of $F_\epsilon$ then implies that for all $t \in \mathbb{N}$, we have (when identifying $\theta^t = \theta_0^t\theta_1^t$ with $(\theta_0^t, \theta_1^t)$ so that $L_\epsilon$ is well-defined on such symbols)

$$\mathbf{x}^{t+1} = aL_\epsilon(\mathbf{x}^t) + (1-a)L_\epsilon(\theta^t). \tag{4}$$

We assume that $\epsilon < \epsilon_a$. Since $\epsilon_a < 1/2$, we can invert this relation to obtain

$$x_s^t = a^{-1}(L_\epsilon^{-1}\mathbf{x}^{t+1})_s + (a-1)a^{-1}\theta_s^t,$$

for all $t \in \mathbb{N}$ and $s \in \{0, 1\}$. Iterating, we have for every $t \in \mathbb{N}$

$$x_s^t = a^{-n}(L_\epsilon^{-n}\mathbf{x}^{t+n})_s + (a-1)\sum_{k=0}^{n-1} a^{-(k+1)}(L_\epsilon^{-k}\theta^{t+k})_s, \tag{5}$$

for all $n \geqslant 1$ and $s \in \{0, 1\}$. Using the norm of $L_\epsilon^{-1}$, it follows that

$$|a^{-n}(L_\epsilon^{-n}\mathbf{x}^{t+n})_s| \leqslant (a(1-2\epsilon))^{-n}\|\mathbf{x}^{t+n}\|,$$

where $s \in \{0, 1\}$ and then $\lim\limits_{n\to+\infty} a^{-n}(L_\epsilon^{-n}\mathbf{x}^{t+n})_s = 0$ for every $(x_0, x_1) \in \mathcal{K}_\epsilon$ if $a(1-2\epsilon) > 1$ i.e. if $\epsilon < \epsilon_a$. In addition, one can check that $(L_\epsilon^{-k}\mathbf{x})_s = \ell_0^{(k)}x_s + \ell_1^{(k)}x_{1-s}$. It results that if the point $(x_0, x_1)$ belongs to $\mathcal{K}_\epsilon$, then by taking the limit $n \to +\infty$ in relation (5), we have for all $t \in \mathbb{N}$ and $s \in \{0, 1\}$

$$x_s^t = (a-1)\sum_{k=0}^{+\infty} a^{-(k+1)}(L_\epsilon^{-k}\theta^{t+k})_s = \psi_\epsilon(\sigma^t R^s\theta).$$

In particular $(x_0, x_1) = \phi_\epsilon(\theta)$ and $\theta \in \mathcal{A}_\epsilon$. Moreover, this relation shows that $F_\epsilon(x_0, x_1) = \mathbf{x}^1 = \phi_\epsilon(\sigma\theta)$, i.e. the conjugacy holds.

Let now $\theta \in \mathcal{A}_\epsilon$ and define $\mathbf{x}(t) = \phi_\epsilon(\sigma^t\theta)$ for all $t \in \mathbb{N}$. By definition of $\phi_\epsilon$, these points satisfy relation (4). These points also satisfy relation (2) because $\theta \in \mathcal{A}_\epsilon$. It results that we have $F_\epsilon(\mathbf{x}(t)) = \mathbf{x}(t+1)$ for every $t$. Moreover, the map $\phi_\epsilon$ is continuous and the set $\Omega$ is compact. Therefore the sequence $\{\mathbf{x}(t)\}$ is a bounded orbit of the CML and thus, $\mathbf{x}(0) \in \mathcal{K}_\epsilon$.

Furthermore, if $\phi_\epsilon(\theta_1) = \phi_\epsilon(\theta_2)$ where $\theta_1, \theta_2 \in \mathcal{A}_\epsilon$, then by relation (3), we have $\theta_1 = \theta_2$. Consequently $\phi_\epsilon$ is a bijection from $\mathcal{A}_\epsilon$ to $\mathcal{K}_\epsilon$. □

## 2.2 Notations and properties of the conjugacy map

We have shown that the CML dynamics in the repeller $\mathcal{K}_\epsilon$ is entirely determined by $\mathcal{A}_\epsilon$. To investigate this set, we need more notations and some properties of the function $\psi_\epsilon$ which are collected in this section.

As usual, concatenations of symbols are called words and words can also be concatenated. Given $\theta \in \Omega$ and $i \leqslant j \in \mathbb{N}$, the notation $(\theta)_i^j = (\theta^i\theta^{i+1}\cdots\theta^j)$ denotes the word composed of symbols from $i$ to $j$.

Superscripts are used to shorten the notation of a $n$-fold concatenation of a symbol or a word. Parenthesis are employed to separate symbols in a word or in a sequence. For instance, $\theta = (01)(10)^n\Delta$ means $\theta^0 = 01$, $\theta^t = 10$ when $t \in \{1, \cdots, n\}$ and $\theta^t = \Delta^{t-n-1}$ when $t > n$ and $\theta = (01)^\infty$ means $\theta^t = 10$ for all $t$.

Moreover, we endow $\Omega$ with the following partial order. Firstly comparing symbols, we say that $\theta_0\theta_1 \leqslant \Delta_0\Delta_1$ if $\theta_0 = 0$ and $\Delta_0 = 1$ or if $\theta_0 = \Delta_0$, $\theta_1 = 1$ and $\Delta_1 = 0$ (in short terms $01 \leqslant 00 \leqslant 11 \leqslant 10$). Now, given two symbolic sequences $\theta, \Delta \in \Omega$, we say that $\theta \leqslant \Delta$ if $\theta^t \leqslant \Delta^t$ for all $t \in \mathbb{N}$.

**Proposition 2.2** *(i) If $\theta \leqslant \Delta$, then $\psi_\epsilon(\theta) \leqslant \psi_\epsilon(\Delta)$.*
*(ii) There exists $k_\epsilon \geqslant 1$ such that if $\theta^t = \Delta^t$ for all $t \neq k, k+1$ for some $k \geqslant 1$ and if*

$$\begin{cases} (\theta)_k^{k+1} = ((10)(11)) \quad \textit{and} \quad (\Delta)_k^{k+1} = ((11)(10)) \quad \textit{if} \quad k \in \{1, \cdots, k_\epsilon\} \\ (\theta)_k^{k+1} = ((11)(10)) \quad \textit{and} \quad (\Delta)_k^{k+1} = ((10)(11)) \quad \textit{if} \quad k > k_\epsilon \end{cases}$$

*then $\psi_\epsilon(\theta) \leqslant \psi_\epsilon(\Delta)$.*

Similarly as in [15], the order on symbolic sequences will allow to determine which sequences are admissible and which are not. But, contrary to as for Lorenz maps, our ordering is only partial and one cannot expect to characterise entirely the set of admissible sequences but rather to obtain approximations or estimates.

*Proof:* To prove (i), we have to show that for all $k \in \mathbb{N}$, we have

$$\eta_k(01) \leqslant \eta_k(00) \leqslant \eta_k(11) \leqslant \eta_k(10),$$

where $\eta_k(\theta_0\theta_1) = \ell_0^{(k)}\theta_0 + \ell_1^{(k)}\theta_1$. We have $\eta_k(01) \leqslant \eta_k(00)$ and $\eta_k(11) \leqslant \eta_k(10)$ because $\ell_1^{(k)} \leqslant 0$ as its definition shows. In addition $\eta_k(00) = 0 < 1 = \ell_0^{(k)} + \ell_1^{(k)} = \eta_k(11)$ and the desired inequality follows.

Property (ii) is equivalent to the following one

$$\begin{cases} \mu_k((10)(11)) \leqslant \mu_k((11)(10)) \quad \text{if} \quad k \in \{1, \cdots, k_\epsilon\} \\ \mu_k((11)(10)) \leqslant \mu_k((10)(11)) \quad \text{if} \quad k > k_\epsilon \end{cases}$$

where $\mu_k(\theta^0\theta^1) = \sum_{i=k}^{k+1} a^{-i}\left(\ell_0^{(i)}\theta_0^{i-k} + \ell_1^{(i)}\theta_1^{i-k}\right)$. Explicit calculations show that

$$\mu_k((11)(10)) - \mu_k((10)(11)) = a^{-k}d(k),$$

where $d(k)$ is a decreasing function of $k$, $d(1) \geqslant 0$ and $d(+\infty) = -\infty$. Property (ii) then immediately follows. $\square$

# 3 Dynamics of piecewise affine expanding CML

In this section, results of the analysis of the set of admissible sequences corresponding to the CML (1) when $\epsilon$ varies in $[0, \epsilon_a)$, together with the consequences for the CML are presented.

## 3.1 Uncoupled domain

Firstly, because of properties of $f$, the complete structure of the uncoupled system is preserved for small coupling [1]. That is to say, we have $\mathcal{A}_\epsilon = \Omega$ when $\epsilon$ is small enough. This property holds for more general maps $f$ than those considered here. But the advantage of dealing with piecewise affine maps is that the largest value of $\epsilon$ for which $\mathcal{A}_\epsilon = \Omega$ can be computed.

**Proposition 3.1** *Let $\iota_a = (a-2)/(2a) > 0$. Every sequence in $\Omega$ is admissible iff $\epsilon \in [0, \iota_a)$.*

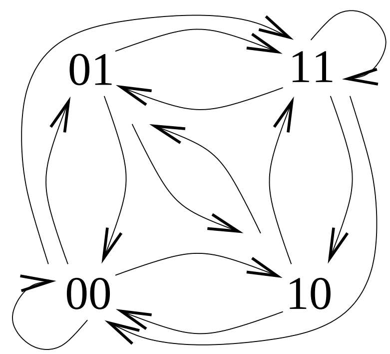


Figure 2: Associated graph to $\Lambda_1$.

By compactness of $\Omega$, together with Proposition 2.1, this result implies that the systems $(\mathcal{K}_\epsilon, F_\epsilon)$ and $(\mathcal{A}_\epsilon, \sigma)$ are topologically conjugated (and hence have the same topological entropy) for every $\epsilon \in [0, \iota_a)$. The domain of couplings where every sequence is admissible was also determined in [6] for piecewise affine CML with infinite number of sites.

*Proof:* Let $\Omega_{(0)} \subset \Omega$ (resp. $\Omega_{(1)} \subset \Omega$) be the subset of sequences $\theta$ such that $\theta_0^0 = 0$ (resp. $\theta_0^0 = 1$). It follows from the admissibility condition that when the following inequalities

$$\sup_{\theta \in \Omega_{(0)}} \psi_\epsilon(\theta) < 1/2 \quad \text{and} \quad \inf_{\theta \in \Omega_{(1)}} \psi_\epsilon(\theta) \geqslant 1/2,$$

hold, every sequence in $\Omega$ is admissible. By continuity of $\psi_\epsilon$ and by compactness of $\Omega$, these bounds are minimum and maximum and thus, these inequalities are also necessary conditions for the admissibility of every symbolic sequence. In addition, the symmetry $\psi_\epsilon(1-\theta) = 1 - \psi_\epsilon(\theta)$ implies that they are equivalent to $\sup\limits_{\theta \in \Omega_{(0)}} \psi_\epsilon(\theta) < 1/2$.

Proposition 2.2-(i) and the fact that $\ell_1^{(0)} = 0$ imply $\sup\limits_{\theta \in \Omega_{(0)}} \psi_\epsilon(\theta) = \psi_\epsilon((0\omega)(10)^\infty)$ where $\omega \in \{0, 1\}$. Moreover explicit calculations show that

$$\psi_\epsilon((0\omega)(10)^\infty) = (a-1) \sum_{k=1}^{+\infty} a^{-(k+1)} \ell_0^{(k)} = \frac{a - 1 - a\epsilon}{a(a - 1 - 2a\epsilon)} < 1/2 \quad \text{iff} \quad \epsilon < \iota_a. \tag{6}$$

$\Box$

Note that relation (6) and the fact that $\ell_1^{(k)} < 0$ for every $k \geqslant 1$ imply that $\mathcal{A}_{\iota_a}$ consists of all sequence $\theta \in \Omega$ excepted those $\theta$ for which there exist $t \in \mathbb{N}$ and $s \in \{0, 1\}$ so that $\sigma^t R^s \theta = (0\omega)(10)^\infty$ for some $\omega \in \{0, 1\}$.

## 3.2 Route to synchronisation

In the domain $\epsilon \in (\iota_a, \epsilon_a)$, we have obtained coupling dependent approximations of the set $\mathcal{A}_\epsilon$ rather than a complete characterisation. The approximations are chosen in a family of sets obtained by restricting the length of words $(10)^k$ and $(01)^k$ but by letting words of 00's and 11's be of arbitrary length.

Precisely, the approximations belong to the family $\{\Omega_n\}$ defined as follows. Firstly, consider the set $\Lambda_n$ of all sequences in $\Omega$ for which the length of every word $(10)^k$ and $(01)^k$ does not exceed $n$. (The graph associated to $\Lambda_1$ is represented Figure 2.) Now, a sequence $\theta \in \Omega$ belongs to $\Omega_n$ if $\theta = (10)^\infty$, if $\theta = (01)^\infty$ or if there exist $k \in \mathbb{N}$ and $\Delta \in \Lambda_n$ such that either $\theta = (01)^k \Delta$ or $\theta = (10)^k \Delta$.

The expression $\psi_\epsilon((0\omega)(10)^\infty)$ in relation (6), together with continuity of the map $\psi_\epsilon$ implies that, for every $\epsilon > \iota_a$, we have $\psi_\epsilon((0\omega)(10)^n\theta) > 1/2$ for every $\theta \in \Omega$ and every $\omega \in \{0, 1\}$ if $n$ is sufficiently large. Therefore, every sequence containing the word $(0\omega)(10)^{n+1}$, or by symmetry the word $(1\omega)(01)^{n+1}$ is not

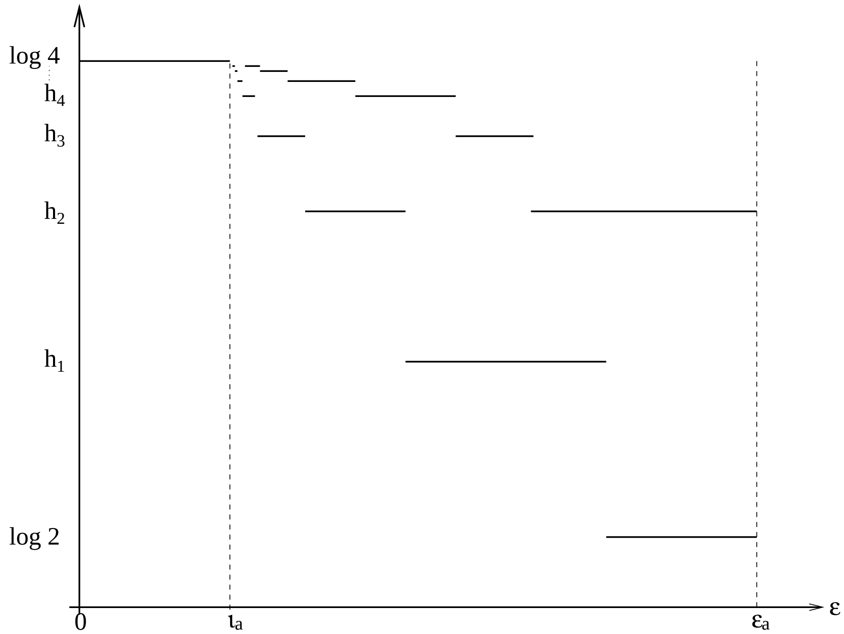


Figure 3: Upper bound $h_{\overline{n}(\epsilon)}$ and lower bound $h_{\underline{n}(\epsilon)}$ for the topological entropy $H_\epsilon$ of the CML versus coupling.

admissible if $n$ is sufficiently large. On the opposite, one can see that if a sequence $\theta \in \Omega_n$ (where $n$ is any integer) contains the word $(0\omega)(10)^k$ or $(1\omega)(01)^k$, then $k \leqslant n$.

The next statement describes the set of admissible sequences in the domain $\epsilon \in (\iota_a, \epsilon_a)$.

**Theorem 3.2** *There exists a right continuous decreasing function* $\underline{n} : (\iota_a, \epsilon_a) \to \mathbb{N}$ *such that* $\lim\limits_{\epsilon \longrightarrow \iota_a} \underline{n}(\epsilon) = +\infty$ *and* $\lim\limits_{\epsilon \longrightarrow \epsilon_a} \underline{n}(\epsilon) = 0$ *and there exists a left continuous decreasing function* $\overline{n} : (\iota_a, \epsilon_a) \to \mathbb{N}$ *such that* $\lim\limits_{\epsilon \longrightarrow \iota_a} \overline{n}(\epsilon) = +\infty$ *and* $\lim\limits_{\epsilon \longrightarrow \epsilon_a} \underline{n}(\epsilon) = 2$*, so that for every* $\epsilon \in (\iota_a, \epsilon_a)$*, we have*

$$\Omega_{\underline{n}(\epsilon)} \subsetneq \mathcal{A}_\epsilon \subset \Omega_{\overline{n}(\epsilon)}.$$

In other words, the dynamics becomes less chaotic and more homogeneous when the coupling increases and this phenomenon is quantified by the bounds $\Omega_{\underline{n}(\epsilon)}$ and $\Omega_{\overline{n}(\epsilon)}$. The decay of $\underline{n}$ and of $\overline{n}$ indicate that the higher the coupling is, the shorter the number of consecutive time steps in the region with symbol 10 and the number of consecutive time steps in the region 01 have to be (excepted when the initial condition belongs to such a region). It means that, when the coupling increases, the orbits spend more time in regions where both coordinates lie on the same side of 1/2, showing that the CML gets closer to the synchronised regime.

(The strict inclusion $\Omega_{\underline{n}(\epsilon)} \subsetneq \mathcal{A}_\epsilon$ implies that $\underline{n}(\epsilon) < \overline{n}(\epsilon)$. However, we believe in the existence of $M \in \mathbb{N}$ such that $\overline{n}(\epsilon) \leqslant \underline{n}(\epsilon) + M$ for every $\epsilon$. One can actually prove the existence of such $M$ for every $a$ such that $\log a \geqslant 1$.)

A consequence of Theorem 3.2 is the existence of upper bound and lower bound for the topological entropy of the CML which are decreasing functions of the coupling. Namely, if $H_\epsilon = h(\mathcal{K}_\epsilon, F_\epsilon)$ is the topological entropy of the CML and if $h_n$ is the topological entropy of the dynamical system $(\Omega_n, \sigma)$, then for all $\epsilon \in (\iota_a, \epsilon_a)$, we have

$$h_{\underline{n}(\epsilon)} \leqslant H_\epsilon \leqslant h_{\overline{n}(\epsilon)}.$$

Indeed, compactness of $\Omega_n$ implies that $\phi(\Omega_{\underline{n}(\epsilon)})$ is compact and then $\phi_\epsilon^{-1}$ is uniformly continuous on this set. Since the topological entropy is invariant by topological conjugacy provided the underlying spaces are compact (see for instance [12]) it results that $h_{\underline{n}(\epsilon)} = h(\phi_\epsilon(\Omega_{\underline{n}(\epsilon)}), F_\epsilon)$ and the left inequality follows. The right inequality is an immediate consequence of conjugacy and uniform continuity of $\phi_\epsilon$.

In Appendix A, we show that the sequence $\{h_n\}$ has the following properties: $h_{n+1} > h_n$ for every $n \in \mathbb{N}$ (and thus $h_{\underline{n}(\epsilon)} < h_{\overline{n}(\epsilon)}$), $h_0 = \log 2$ and $\lim\limits_{n\to+\infty} h_n = \log 4$. Together with Proposition 3.1, these properties allow to construct the graphs of $\epsilon \mapsto h_{\underline{n}(\epsilon)}$ and of $\epsilon \mapsto h_{\overline{n}(\epsilon)}$, see Figure 3.

A complementary result to Theorem 3.2 is monotonicity of the admissibility of symbolic sequences with coupling. The next statement tells us that this assertion holds in a neighbourhood of $\iota_a$.

**Proposition 3.3** *There exists $\eta_a > \iota_a$ such that for every $\epsilon_1 < \epsilon_2 \in [0, \eta_a)$, we have $\mathcal{A}_{\epsilon_1} \supset \mathcal{A}_{\epsilon_2}$.*

The proofs of results in this section are organised as follows. In the next section the existence of the function $\underline{n}$ is shown. The existence of $\overline{n}$ is proved in Section 5. The proof of Proposition 3.3 is given in Appendix B.

# 4 Existence of the function $\underline{n}$

The explicit expression of $\psi_\epsilon$ and its properties allow to show that, for every $n \in \mathbb{N}$, there exists a critical coupling below which every sequence in $\Omega_n$ satisfies the admissibility condition. Precisely, we have the following statement.

**Proposition 4.1** *There exists a strictly decreasing sequence $\{\lambda_{a,n}\}_{n\in\mathbb{N}}$ with $\lambda_{a,0} = \epsilon_a$ and $\lim\limits_{n\to+\infty} \lambda_{a,n} = \iota_a$ such that, for every $n \in \mathbb{N}$, we have $\Omega_n \subset \mathcal{A}_\epsilon$ iff $\epsilon < \lambda_{a,n}$.*

The existence of the lower bound $\underline{n}$ in Theorem 3.2 as desired is a immediate consequence of this statement. Namely, we have for every $\epsilon \in (\iota_a, \epsilon_a)$

$$\underline{n}(\epsilon) = \max\{n \in \mathbb{N} \ : \ \epsilon < \lambda_{a,n}\},$$

and this function is decreasing, right continuous and has range $\mathbb{N}$. The proof that the inclusion $\Omega_{\underline{n}(\epsilon)} \subset \mathcal{A}_\epsilon$ is strict is given in section 4.3.

Proposition 4.1 is proved using the same arguments as those in the proof of Proposition 3.1. Firstly, using properties of $\psi_\epsilon$ and since each set $\Omega_n$ is invariant under $\sigma$, under $R$ and under the change $\theta \mapsto 1-\theta$, one proves that every sequence in $\Omega_n$ is admissible iff $\sup\limits_{\theta\in\Omega_n \ : \ \theta_0^0=0} \psi_\epsilon(\theta) < 1/2$. Secondly, the partial order on $\Omega$ allows to prove that the previous supremum is reached for the maximal sequence in $\Omega_n$ with respect to the lexicographic order.

**Lemma 4.2** *For every $n \in \mathbb{N}$, we have* $\sup\limits_{\theta\in\Omega_n \ : \ \theta_0^0=0} \psi_\epsilon(\theta) = \psi_\epsilon((00)((10)^n(11))^\infty)$.

Then, studying the behaviour of this maximum with coupling gives the existence of the $\lambda_{a,n}$'s and their properties.

**Lemma 4.3** *There exists a strictly decreasing sequence $\{\lambda_{a,n}\}_{n\in\mathbb{N}}$ with $\lambda_{a,0} = \epsilon_a$ and $\lim\limits_{n\to+\infty} \lambda_{a,n} = \iota_a$ such that, for every $n \in \mathbb{N}$, we have $\psi_\epsilon((00)((10)^n(11))^\infty) < 1/2$ iff $\epsilon < \lambda_{a,n}$.*

The proofs of Lemmas 4.2 and 4.3 are given in sections below.

## 4.1 Proof of Lemma 4.2

By using the partial order, we reduce successively the set where to find the supremum $\sup\limits_{\theta\in\Omega_n \ : \ \theta_0^0=0} \psi_\epsilon(\theta)$.

Firstly, recall the subset $\Lambda_n \subset \Omega_n$ of symbolic sequences for which the length of every word $(10)^k$ and $(01)^k$ is not larger than $n$. Let $\theta \in \Omega_n \backslash \Lambda_n$ be such that $\theta_0^0 = 0$. Consider the sequence $\bar{\theta}$ obtained by changing

every symbol 01 in $\theta$ by 00. The sequence $\bar{\theta} \in \Lambda_n$ and by Proposition 2.2-(i), we have $\psi_\epsilon(\theta) \leqslant \psi_\epsilon(\bar{\theta})$. These arguments allow to show that

$$\sup_{\theta \in \Omega_n \ : \ \theta_0^0 = 0} \psi_\epsilon(\theta) = \sup_{\theta \in \Lambda_n \ \ \theta_0^0 = 0} \psi_\epsilon(\theta).$$

Since $\Lambda_0 = \{00, 11\}^{\mathbb{N}}$, the result of Lemma 4.2 immediately follows from Proposition 2.2-(i) in the case where $n = 0$, i.e. $\sup\limits_{\theta \in \Lambda_0 \ : \ \theta_0^0 = 0} \psi_\epsilon(\theta) = \psi_\epsilon((00)(11)^\infty)$. Therefore, in the rest of the proof, we assume that $n \geqslant 1$.

In a second step, we restrict to a set of sequences whose symbols, for positive times, are either 10 or 11. This set $P_n$ is the subset of sequences $\theta \in \Lambda_n$ with $\theta_0^0 = 0$ such that, for all $t \geqslant 1$, $\theta^t \in \{11, 10\}$ and if $\theta^t = 11$, there exists $i \in \{0, \cdots, \min\{n, t-1\}\}$ so that $(\theta)_{t-i}^{t-i+n} = (10)^i(11)(10)^{n-i}$. Note that changing a symbol 11 by 10 in an element of $P_n$ creates a sequence which does not belong to $\Lambda_n$. Thus, $P_n$ is the smallest subset of $\Lambda_n$ which can be obtained by only using Proposition 2.2-(i) when restricting and we have the following statement.

**Lemma 4.4** *For every $n \geqslant 1$, we have* $\sup\limits_{\theta \in \Lambda_n \ : \ \theta_0^0 = 0} \psi_\epsilon(\theta) = \sup\limits_{\theta \in P_n} \psi_\epsilon(\theta)$.

*Proof:* As before, we show that for every $\theta \in \Lambda_n$ such that $\theta_0^0 = 0$, there exists $\bar{\theta} \in P_n$ such that $\psi_\epsilon(\theta) \leqslant \psi_\epsilon(\bar{\theta})$.

If $\theta \in \Lambda_n$ and $\theta_0^0 = 0$ but $\theta \notin P_n$, then there exists $t \geqslant 1$ such that $\theta^t \neq 10$ and $(\theta)_{t-i}^{t-i+n} \neq (10)^i(11)(10)^{n-i}$ for all $i \in \{0, \ldots, \min\{n, t-1\}\}$. Let $t(\theta)$ be the smallest of such integers.

Replace the symbol $\theta^{t(\theta)}$ by 10 if this change does not create a word $(10)^{n+1}$ in the resulting sequence, by 11 otherwise. By Proposition 2.2-(i), the resulting sequence, say $A(\theta)$ (belongs to $\Lambda_n$ and) is such that $\psi_\epsilon(\theta) \leqslant \psi_\epsilon(A(\theta))$ and $t(A(\theta)) > t(\theta)$.

By iterating, we conclude that, for every $\theta \in \Lambda_n$ with $\theta_0^0 = 0$, we have $\psi_\epsilon(\theta) \leqslant \psi_\epsilon(\bar{\theta})$ where $\bar{\theta} := \lim\limits_{k \to +\infty} A^k(\theta)$ (the limit exists because $t(A(\theta)) > t(\theta)$). Because $t(A(\theta)) > t(\theta)$, we have $t(\bar{\theta}) = +\infty$ and hence $\bar{\theta} \in P_n$. $\square$

To restrict further the set on which the supremum is reached, we now use Proposition 2.2-(ii) in a similar reasoning. The resulting set is the finite set $Q_n$ defined by

$$Q_n = \{(0\omega)(10)^i((11)(10)^n)^\infty \ : \ i \in \{0, \cdots, n\}, \ \omega \in \{0, 1\}\}$$

**Lemma 4.5** *For every $n \geqslant 1$, we have* $\sup\limits_{\theta \in P_n} \psi_\epsilon(\theta) = \max\limits_{\theta \in Q_n} \psi_\epsilon(\theta)$.

The proof is given in Appendix C. Finally, we show that the maximum is reached for the maximum in the lexicographic order.

**Lemma 4.6** *For every $n \geqslant 1$, we have* $\max\limits_{\theta \in Q_n} \psi_\epsilon(\theta) = \psi_\epsilon((00)((10)^n(11))^\infty)$.

*Proof:* We show that, for every $i \in \{0, \cdots, n-1\}$, we have

$$\psi_\epsilon((00)(10)^i((11)(10)^n)^\infty) \leqslant \psi_\epsilon((00)((10)^n(11))^\infty).$$

By explicit calculations, one obtains that these inequalities are equivalent to the following ones

$$\sum_{p=0}^{+\infty} a^{-(i+1+p(n+1))} \ell_1^{((i+1+p(n+1)))} \leqslant \sum_{p=0}^{+\infty} a^{-(p+1)(n+1)} \ell_1^{((p+1)(n+1))},$$

where $i \in \{0, \cdots, n-1\}$. Using the definition of $\ell_1^{(k)}$, these in turn simplify to

$$\sum_{k=0}^{n-i-1} a^k \sum_{j=n-i}^{n} b_\epsilon^j \leqslant \sum_{k=0}^{n-i-1} b_\epsilon^k \sum_{j=n-i}^{n} a^j, \quad i \in \{0, \cdots, n-1\}, \tag{7}$$

where $b_\epsilon = a(1-2\epsilon)$. Since $1 < b_\epsilon \leqslant a$ when $\epsilon \in [0, \epsilon_a)$, for every $j \in \{n-i, \cdots, n\}$ the inequality $\sum\limits_{k=0}^{n-i-1} a^{k-j} \leqslant \sum\limits_{k=0}^{n-i-1} b_\epsilon^{k-j}$ holds, i.e. we have $\sum\limits_{k=0}^{n-i-1} a^k b_\epsilon^j \leqslant \sum\limits_{k=0}^{n-i-1} b_\epsilon^k a^j$ which implies (7). $\square$

## 4.2 Proof of Lemma 4.3

The quantity $\psi_\epsilon((00)(11)^\infty) < 1/2$ because $a > 2$. In other words, every sequence in $\Omega_0$ is admissible for every $\epsilon < \epsilon_a$ and $\lambda_{a,0} = \epsilon_a$.

Let now $n \geqslant 1$ be fixed. We show that the map $\epsilon \mapsto \psi_\epsilon((00)((10)^n(11))^\infty)$ is strictly increasing and diverges at $\epsilon_a$. Together with the fact that

$$\psi_{\iota_a}((00)((10)^n(11))^\infty) < \psi_{\iota_a}((00)(10)^\infty) = 1/2,$$

this implies the existence of $\lambda_{a,n} \in (\iota_a, \epsilon_a)$.

We have

$$\psi_\epsilon((00)((10)^n(11))^\infty) = \frac{a-1}{2a}\left(\frac{1}{a-1} + \frac{1}{b_\epsilon - 1} + \frac{1}{a^{n+1}-1} - \frac{1}{b_\epsilon^{n+1}-1}\right),$$

where again $b_\epsilon = a(1-2\epsilon) > 1$. Hence $\lim\limits_{\epsilon \to \epsilon_a} \psi_\epsilon((00)((10)^n(11))^\infty) = +\infty$ because $b_\epsilon \to 1$ when $\epsilon \to \epsilon_a$. Moreover, explicit calculations shows that the sign of the derivative $\psi'_\epsilon((00)((10)^n(11))^\infty)$ is the same as the sign of

$$(b_\epsilon^{n+1}-1)^2 - (n+1)b_\epsilon^n(b_\epsilon - 1)^2 = (b_\epsilon - 1)^2\left(\left(\sum_{k=0}^n b_\epsilon^k\right)^2 - (n+1)b_\epsilon^n\right),$$

and the latter is positive because $b_\epsilon > 0$.

To prove the inequality $\lambda_{a,n+1} < \lambda_{a,n}$, $n \in \mathbb{N}$, it suffices to show that

$$\psi_\epsilon((00)((10)^n(11))^\infty) < \psi_\epsilon((00)((10)^{n+1}(11))^\infty).$$

Using the expression of $\psi_\epsilon((00)((10)^n(11))^\infty)$, the latter is equivalent to

$$\frac{1}{b_\epsilon^{n+1}-1} - \frac{1}{b_\epsilon^n - 1} < \frac{1}{a^{n+1}-1} - \frac{1}{a^n - 1},$$

where $n \geqslant 1$, and hence equivalent to the property that the the map $a \mapsto (a^{n+1}-1)^{-1} - (a^n-1)^{-1}$ is strictly increasing when $a > 1$. Computing its derivative, we obtain that this condition holds iff $n(\sum\limits_{k=0}^n a^k)^2 > (n+1)a(\sum\limits_{k=0}^{n-1} a^k)^2$, which holds for every $a > 1$ and $n \geqslant 1$.

Each $\lambda_{a,n}$ can be defined as $\sup\{\epsilon : \psi_\epsilon((00)((10)^n(11))^\infty) < 1/2\}$. The sequence $\{\lambda_{a,n}\}$ is decreasing and bounded below. Therefore it converges to a limit which, by continuity of $\psi_\epsilon$, is equal to $\sup\{\epsilon : \psi_\epsilon((00)(10)^\infty) < 1/2\} = \iota_a$.

## 4.3 Proof of the strict inclusion $\Omega_{\underline{n}(\epsilon)} \subsetneq \mathcal{A}_\epsilon$

To that goal, we prove the existence, for every $n \geqslant 2$, of a sequence in the complement set of $\Omega_n$ which belongs to $\mathcal{A}_\epsilon$ for every $\epsilon < \lambda_{a,n}$ where $\lambda_{a,n}$ is defined in Lemma 4.3. The desired sequence is the periodic sequence $((01)^{n+1}(10)^{n+1})^\infty$.

Indeed by applying the maps $\sigma$ and $\theta \mapsto 1-\theta$, a necessary and sufficient condition for $((01)^{n+1}(10)^{n+1})^\infty \in \mathcal{A}_\epsilon$ is

$$\max_{p \in \{1,\cdots,n+1\}} \psi_\epsilon((01)^p((10)^{n+1}(01)^{n+1})^\infty) < 1/2.$$

Recall now the function $\mu_k$ employed in the proof of Proposition 2.2-(ii). It can be shown that the condition $\mu_k((01)(10)) < \mu_k((10)(01))$ for all $k \in \mathbb{N}$ holds for every $\epsilon < \epsilon_a$. Using this property repeatedly, we obtain

$$\begin{aligned}
\psi_\epsilon((01)^p((10)^{n+1}(01)^{n+1})^\infty) &< \psi_\epsilon((01)^{p-1}((10)(01)(10)^n(01)^n)^\infty) \\
&< \psi_\epsilon((01)^{p-1}((10)^2(01)(10)^{n-1}(01)^n)^\infty) \\
&< \cdots \\
&< \psi_\epsilon((01)^{p-1}((10)^{n+1}(01)^{n+1})^\infty)
\end{aligned}$$

showing that the previous maximum is reached for $p = 1$. Moreover, using the property $\mu_k((10)(01)) < \mu_k((11)(10))$ for all $k \in \mathbb{N}$, which holds for every $\epsilon \in (\iota_a, \epsilon_a)$, a similar argument shows that

$$\psi_\epsilon((01)((10)^{n+1}(01)^{n+1})^\infty) < \psi_\epsilon((01)((10)^n(11)(10)(01)^n)^\infty).$$

Since $(01)((10)^n(11)(10)(01)^n)^\infty \in \Omega_n$, this inequality implies that

$$\psi_\epsilon((01)((10)^{n+1}(01)^{n+1})^\infty) < 1/2,$$

for every $\epsilon \in (\iota_a, \lambda_{a,n})$ and thus $\Omega_{\underline{n}(\epsilon)} \subsetneq \mathcal{A}_\epsilon$.

# 5 Existence of the function $\overline{n}$

As for the lower bound, to obtain the upper bound $\overline{n}$ in Theorem 3.2, we prove the existence of values of the coupling above which every sequence in the complement set of $\Omega_n$ does not satisfy the admissibility condition. The main statement of this section is the following one.

**Proposition 5.1** *There exists a decreasing sequence $\{v_{a,n}\}_{n\geqslant 2}$ with $v_{a,n} < \epsilon_a$ and $\lim\limits_{n\to+\infty} v_{a,n} = \iota_a$ such that, for every $n \geqslant 2$, we have $\mathcal{A}_\epsilon \subset \Omega_n$ if $\epsilon \in (v_{a,n}, \epsilon_a)$.*

Similarly as for the lower bound, Proposition 5.1 implies the existence of the function $\overline{n}$ with the desired properties. This function is defined by

$$\overline{n}(\epsilon) = \min\{n \geqslant 2 \ : \ v_{a,n} < \epsilon\},$$

for every $\epsilon \in (\iota_a, \epsilon_a)$.

The conclusion of Proposition 5.1 does not hold for $n = 0$, i.e. we have $v_{a,0} = \epsilon_a$. This is because explicit calculations show that the sequence $((01)(10))^\infty$ is admissible for every $\epsilon \in [0, \epsilon_a)$. We do not know if this statement holds for $n = 1$.

Let $V_n$ be the complement set of $\Omega_n$ in $\Omega$. The proof of Proposition 5.1 uses the following sufficient condition for every sequence in $V_n$ not to belong to $\mathcal{A}_\epsilon$: for every $\theta \in \Omega$, either we have $\psi_\epsilon((0\omega)(10)^{n+1}\theta) \geqslant 1/2$ or we have $\psi_\epsilon((10)\theta) < 1/2$.

(Note that the following sufficient condition

$$\inf_{\theta\in\Omega} \psi_\epsilon((00)(10)^{n+1}\theta) = \psi_\epsilon((00)(10)^{n+1}(01)^\infty) \geqslant 1/2,$$

does not hold when $\epsilon$ is close to $\epsilon_a$ because $\lim\limits_{\epsilon\to\epsilon_a} \psi_\epsilon((00)(10)^{n+1}(01)^\infty) = -\infty$. Therefore, it cannot be used to prove the existence of $v_{a,n}$. The same comment applies to the improved condition

$$\psi_\epsilon((00)(10)^{n+1}((01)^{n+1}(00))^\infty) \geqslant 1/2.)$$

To employ this sufficient condition, we need some notations and an auxiliary result. Let the functions $S(\theta) = (2a)^{-1}(a-1)\sum\limits_{k=0}^{+\infty} a^{-k}(\theta_0^k + \theta_1^k)$ and $D_\epsilon(\theta) = (2a)^{-1}(a-1)\sum\limits_{k=0}^{+\infty} b_\epsilon^{-k}(\theta_0^k - \theta_1^k)$ which is well-defined when $\epsilon < \epsilon_a$. Recall that $b_\epsilon = a(1-2\epsilon)$ decreases in $(1,2)$ when $\epsilon$ increases in $(\iota_a, \epsilon_a)$.

Using the definitions of $\psi_\epsilon$ and of $\ell_0^{(k)}$ and $\ell_1^{(k)}$ we obtain for every $\omega \in \{0,1\}$

$$\psi_\epsilon((0\omega)(10)^{n+1}\theta) = (2a)^{-1}(a-1)\sum_{k=1}^{n+1}(a^{-k} + b_\epsilon^{-k}) + a^{-(n+2)}S(\theta) + b_\epsilon^{-(n+2)}D_\epsilon(\theta),$$

and $\psi_\epsilon((10)\theta) = a^{-1}(a-1) + a^{-1}S(\theta) + b_\epsilon^{-1}D_\epsilon(\theta)$. Consider also the positive quantity $M_{a,n} = a^{-1}(a-1) + 2^{n+1}a^{-(n+2)}$ and the following statement.

**Lemma 5.2** *For every $n \geqslant 1$, there exists $\rho_{a,n} < \epsilon_a$ such that for every $\epsilon \in (\rho_{a,n}, \epsilon_a)$ and every $\theta \in \Omega$ so that $D_\epsilon(\theta) \in [-1/2, M_{a,n}]$, we have $\psi_\epsilon((10)\theta) < \psi_\epsilon((0\omega)(10)^{n+1}\theta)$.*

*Proof of Lemma 5.2:* Using the notations above, one shows that the inequality $\psi_\epsilon((10)\theta) < \psi_\epsilon((0\omega)(10)^{n+1}\theta)$ is equivalent to the following one

$$b_\epsilon^{-1}(1 - b_\epsilon^{-(n+1)})D_\epsilon(\theta) + (2a)^{-1}(a-1)(n+1-\sum_{k=1}^{n+1} b_\epsilon^{-k})$$
$$< (2a)^{-1}(a-1)(\sum_{k=1}^{n+1} a^{-k} + n - 1) - a^{-1}(1 - a^{-(n+1)})S(\theta).$$

The left hand side equals 0 when $b = 1$. Using the inequality $S(\theta) \leqslant 1$, one proves that for every $a > 2$ and $n \geqslant 2$, the following inequality holds (and hence the previous inequality holds for $\epsilon = \epsilon_a$ if $|D_{\epsilon_a}(\theta)| < +\infty$)

$$(2a)^{-1}(a-1)(\sum_{k=1}^{n+1} a^{-k} + n - 1) - a^{-1}(1 - a^{-(n+1)})S(\theta) > 0.$$

Since the assumption on $\theta$ forces $|D_\epsilon(\theta)| \leqslant \max\{1/2, |M_{a,n}|\}$ and since this bound does not depend on $\epsilon$, by continuity with $\epsilon$, there exists $\rho_{a,n} < \epsilon_a$ and so that for every $\epsilon \in (\rho_{a,n}, \epsilon_a)$, we have $\psi_\epsilon((10)\theta) < \psi_\epsilon((0\omega)(10)^{n+1}\theta)$ for every $\theta \in \Omega$ such that $D_\epsilon(\theta) \in [-1/2, M_{a,n}]$. □

*Proof of Proposition 5.1:* Let $\bar{v}_{a,n} = \max\{\iota_a, \rho_{a,n}\}$ and assume that $\epsilon \in (\bar{v}_{a,n}, \epsilon_a)$. Given any sequence $\Delta \in V_n$, there exists $(s,t)$ such that the sequence $\sigma^t R^s \Delta$ either equals $(0\omega)(10)^{n+1}\theta$ or equals $(1\omega)(01)^{n+1}\theta$ for some $\theta \in \Omega$ and $\omega \in \{0,1\}$. Assume that $\sigma^t R^s \Delta = (0\omega)(10)^{n+1}\theta$, the other case can be completed by using the symmetry $\psi_\epsilon(1-\theta) = 1 - \psi_\epsilon(\theta)$. One of the following conditions holds.

(a) $D_\epsilon(\theta) > M_{a,n}$,

(b) $D_\epsilon(\theta) \in [-1/2, M_{a,n}]$,

(c) $D_\epsilon(\theta) < -1/2$.

In case (a), using the inequalities $S(\theta) \geqslant 0$ and $b_\epsilon < 2$ in the expression of $\psi_\epsilon((0\omega)(10)^{n+1}\theta)$ and using the definition of $M_{a,n}$, we obtain for every $\epsilon > \iota_a$,

$$\psi_\epsilon((0\omega)(10)^{n+1}\theta) > (2a)^{-1}(a-1)\sum_{k=1}^{n+1}(a^{-k} + 2^{-k}) + 2^{-(n+2)}M_{a,n} = 1/2,$$

showing that $\Delta$ does not belong to $\mathcal{A}_\epsilon$. Similarly, in case (c), since $S(\theta) \leqslant 1$, using the expression of $\psi_\epsilon((10)\theta)$, we obtain

$$\psi_\epsilon((10)\theta) \leqslant 1 + D_\epsilon(\theta) < 1/2,$$

which gives the same conclusion. Finally in case (b), one can apply Lemma 5.2 to conclude that either $\psi_\epsilon((0\omega)(10)^{n+1}\theta) \geqslant 1/2$ or $\psi_\epsilon((10)\theta) < 1/2$. In all cases, we have shown that the original sequence $\Delta$ does not belong to $\mathcal{A}_\epsilon$.

We have shown that no sequence in $V_n$ belongs to $\mathcal{A}_\epsilon$ when $\epsilon \in (\bar{v}_{a,n}, \epsilon_a)$. Let $v_{a,n}$ be the infimum of couplings $\epsilon$ such that no sequence in $V_n$ belongs to $\mathcal{A}_\epsilon$ for all $\epsilon \in (v_{a,n}, \epsilon_a)$. By Proposition 3.1, we have $v_{a,n} \geqslant \iota_a$ for every $n \geqslant 2$. Moreover, since $V_{n+1} \subset V_n$, we have $v_{a,n+1} \leqslant v_{a,n}$ and thus, the limit $v_a := \lim\limits_{n\to+\infty} v_{a,n}$ exists. Finally, continuity of $\psi_\epsilon$ implies that $v_a = \iota_a$. □

# 6 Dynamics of coupled piecewise increasing maps lattices

As already said, the existence of a conjugacy between the weakly coupled system and the uncoupled system has been proved for more general CML than only piecewise affine ones. In this section, we show that the results of Theorem 3.2 also extend to more general CML.

These CML are defined as in (1) but instead of the piecewise affine map, we consider any map $f : \mathbb{R} \to \mathbb{R}$ which is continuous and increasing on $(-\infty, 1/2)$ and on $[1/2, +\infty)$ and which is such that there exists $a_f > 1$ so that for all $x, y \in \mathbb{R}$, we have

$$|f(x) - f(y)| \geqslant a_f |x - y|. \tag{8}$$

In the sequel, the notation $a_f$ will always means the largest of such numbers and $F_{f,\epsilon}$ will denote the CML defined using such a map $f$.

## 6.1 Symbolic description

As for the piecewise affine system, the first step of analysis of $F_{f,\epsilon}$ is to show that, due to expansiveness (8), the system $(\mathcal{K}_{f,\epsilon}, F_{f,\epsilon})$ (where $\mathcal{K}_{f,\epsilon}$ still denote the repeller of the CML) admits a symbolic description. Let $\epsilon_f = (a_f - 1)/(2a_f) > 0$.

**Proposition 6.1** *For every $\epsilon \in [0, \epsilon_f)$, there exists a map $\psi_{f,\epsilon} : \Omega \to \mathbb{R}$ such that $\mathcal{K}_{f,\epsilon} = \phi_{f,\epsilon}(\mathcal{A}_{f,\epsilon})$ where $\phi_{f,\epsilon}(\theta) = (\psi_{f,\epsilon}(\theta), \psi_{f,\epsilon}(R\theta))$ and where $\mathcal{A}_{f,\epsilon}$ is the set of sequences $\theta$ which satisfy the admissibility condition*

$$\theta_s^t = H(\psi_{f,\epsilon}(\sigma^t R^s \theta) - 1/2),$$

*for all $t \in \mathbb{N}$ and $s \in \{0,1\}$. Moreover, the map $\phi_{f,\epsilon}$ is uniformly continuous, one-to-one and conjugates the systems $(\mathcal{K}_{f,\epsilon}, F_{f,\epsilon})$ and $(\mathcal{A}_{f,\epsilon}, \sigma)$.*

*Proof:* Let $\epsilon \in [0, \epsilon_f)$ and let $(x_0, x_1)$ be a point in $\mathcal{K}_{f,\epsilon}$. Let $f_0$ (resp. $f_1$) be the restriction of $f$ to $(-\infty, 1/2)$ (resp. $[1/2, +\infty)$). The inequality $\epsilon_f < 1/2$ ensures that $L_\epsilon$ is invertible. The definition of $F_{f,\epsilon}$ then shows that the following relation holds between the iterates of the orbit $\{\mathbf{x}^t\}$ issued from $(x_0, x_1)$

$$x_s^t = (G_{\theta^t}(\mathbf{x}^{t+1}))_s,$$

for all $t \in \mathbb{N}$ and $s \in \{0,1\}$. Here $\theta^t = \theta_0^t \theta_1^t$ where $\theta_s^t = H(x_s^t - 1/2)$ are the components of the code $\theta$ and the map $G_{\theta^0}$ is defined by

$$G_{\theta^0}(\mathbf{x}) = (f_{\theta_0^0}^{-1}((L_\epsilon^{-1}\mathbf{x})_0), f_{\theta_1^0}^{-1}((L_\epsilon^{-1}\mathbf{x})_1)). \tag{9}$$

By iterating it results that

$$x_s^t = (G_{\theta^t} \circ G_{\theta^{t+1}} \circ \cdots \circ G_{\theta^{t+n}}(\mathbf{x}^{t+n+1}))_s.$$

Let us prove that these components only depend on $\sigma^t\theta$, i.e. on $\{\theta^n\}_{n \geqslant t}$. Firstly, given $t$, the sequence $\{G_{\theta^t} \circ G_{\theta^{t+1}} \circ \cdots \circ G_{\theta^{t+n}}(\mathbf{x}^{t+n+1})\}_{n\in\mathbb{N}}$ is constant, so its limit exists. Moreover, relation (8) and $\|L_\epsilon^{-1}\| = (1-2\epsilon)^{-1}$ imply the inequality

$$\|G_{\theta^0}(\mathbf{x}) - G_{\theta^0}(\mathbf{y})\| \leqslant (a_f(1-2\epsilon))^{-1}\|\mathbf{x} - \mathbf{y}\|. \tag{10}$$

Since $(a_f(1-2\epsilon))^{-1} < 1$ when $\epsilon < \epsilon_f$, it results that

$$\lim_{n\to+\infty} \|G_{\theta^t} \circ G_{\theta^{t+1}} \circ \cdots \circ G_{\theta^{t+n}}(\mathbf{x}^{t+n+1}) - G_{\theta^t} \circ G_{\theta^{t+1}} \circ \cdots \circ G_{\theta^{t+n}}(\mathbf{y}^{t+n+1})\| = 0,$$

if the orbits $\{\mathbf{x}^t\}$ and $\{\mathbf{y}^t\}$ are bounded and have the same code. Consequently, the limit only depends on the sequence $\sigma^t\theta$, that is to say $x_s^t = \tilde{\psi}(\sigma^t\theta, s)$. Moreover, the relation (9) shows that $\tilde{\psi}(R\theta, s) = \tilde{\psi}(\theta, 1-s)$. Denoting by $\psi_{f,\epsilon}(\theta)$ the quantity $\tilde{\psi}(\theta, 0)$, we conclude that every point in $(x_0, x_1) \in \mathcal{K}_{f,\epsilon}$ writes $(x_0, x_1) = \phi_{f,\epsilon}(\theta)$, that the sequence $\theta \in \mathcal{A}_{f,\epsilon}$ and that $F_{f,\epsilon}(\phi_{f,\epsilon}(\theta)) = \phi_{f,\epsilon}(\sigma\theta)$.

Now, to prove that every $\theta \in \mathcal{A}_{f,\epsilon}$ codes a point in $\mathcal{K}_{f,\epsilon}$, we first linearly extend the monotonic components of $f$ over the whole $\mathbb{R}$, i.e. we set $f_0(x) = a_f(x - 1/2) + f_0(1/2 - 0)$ for $x \geqslant 1/2$ and $f_1(x) = a_f(x-1/2) + f_1(1/2)$ if $x < 1/2$. This allows to define the image $\psi_{f,\epsilon}(\theta)$ for every $\theta \in \Omega$ as the following limit

$$\psi_{f,\epsilon}(\theta) := \lim_{t\to+\infty} (G_{\theta^0} \circ \cdots \circ G_{\theta^t}(\mathbf{x}))_0.$$

Indeed the choice of extensions for the monotonic components of $f$ ensures that contraction (10) holds for every pair $\mathbf{x}, \mathbf{y}$ of points in $\mathbb{R}^2$ and every symbol. It results that if it exists, the previous limit does not depend on $\mathbf{x} \in \mathbb{R}^2$.

To show that the limit exists, we note that the monotony and expansiveness of the $f_s$'s imply the existence of $C > 0$ such that $|f_s^{-1}(x)| \leqslant a_f^{-1}|x| + C$ for every $x \in \mathbb{R}$. Consequently, we have

$$\|G_{\theta^0}(\mathbf{x})\| \leqslant (a_f(1-2\epsilon))^{-1}\|\mathbf{x}\| + C(1-2\epsilon)^{-1},$$

and then the sequence $\{G_{\theta^0} \circ \cdots \circ G_{\theta^t}(\mathbf{x})\}$ is bounded. This property, together with the contraction (10), allows to prove that this sequence is a Cauchy sequence and hence has a limit in $\mathbb{R}^2$.

Once $\psi_{f,\epsilon}(\theta)$ has been defined for every symbolic sequence, one can develop similar arguments to those in the proof of Proposition 2.1 to show that if $\theta_s^t = H(\psi_{f,\epsilon}(\sigma^t R^s\theta) - 1/2)$ for every $s, t$, then $\{\phi_{f,\epsilon}(\sigma^t\theta)\}$ is a bounded orbit of $F_{f,\epsilon}$. That is to say, we have $\phi_{f,\epsilon}(\mathcal{A}_{f,\epsilon}) \subset \mathcal{K}_\epsilon$.

Finally, one easily proves that the map $\phi_{f,\epsilon}$ is one-to-one and uniformly continuous and the proof is complete. □

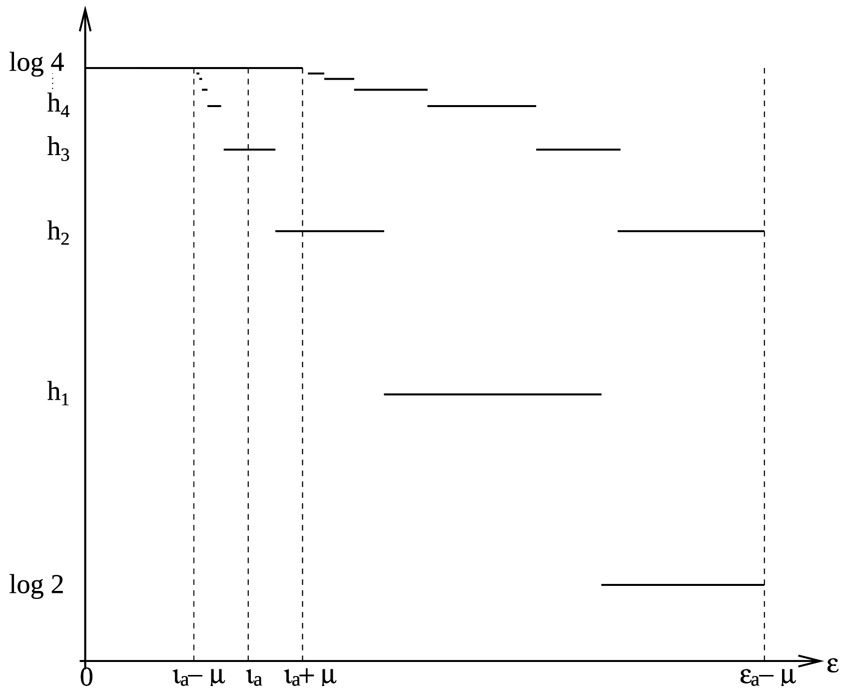


Figure 4: Upper bound and lower bound for the topological entropy of $(\mathcal{K}_{f,\epsilon}, F_{f,\epsilon})$ versus coupling when $f$ is such that $a_f \in (a_\mu, a)$ and $\|f - f_a\| < \delta$ (see Theorem 6.2).

## 6.2 Approximations of symbolic systems

We now show that, up to arbitrary shifts of the functions $\underline{n}$ and $\overline{n}$, the approximations obtained for the piecewise affine map also hold for the symbolic dynamics of CML with local map sufficiently close to the piecewise affine one. This means that small perturbations of the local map result in small perturbations of (approximations of) the complete dynamical picture when the coupling increases and then in bounds on the function “topological entropy versus coupling”.

We shall denote by $f_a$ the piecewise affine map $x \mapsto ax + (1-a)H(x)$. The existence of approximations for more general CML and their continuous dependence with the local map in the $C^1$-topology is given in the following statement.

**Theorem 6.2** *For every $\mu > 0$ sufficiently small, there exists $a_\mu \in (1, a)$ such that for every $\alpha \in (a_\mu, a)$, there exists $\delta > 0$ so that every piecewise increasing expanding map $f$ with $a_f = \alpha$ and $\|f - f_a\| < \delta$ is such that $\epsilon_f > \epsilon_a - \mu$ and satisfies the following properties:*

*(i) $\mathcal{A}_{f,\epsilon} = \Omega$ if $\epsilon \in [0, \iota_a - \mu)$ and $\mathcal{A}_{f,\epsilon} \subsetneq \Omega$ if $\epsilon \in (\iota_a + \mu, \epsilon_a - \mu)$,*

*(ii) $\Omega_{\underline{n}^\mu(\epsilon)} \subsetneq \mathcal{A}_{f,\epsilon}$ for every $\epsilon \in (\iota_a - \mu, \epsilon_a - \mu)$ where $\underline{n}^\mu$ is a right continuous decreasing function with range $\mathbb{N}$ satisfying $\underline{n}^\mu(\epsilon) \geqslant \underline{n}(\epsilon + \mu)$ for every $\epsilon \in (\iota_a - \mu, \epsilon_a - \mu)$,*

*(iii) $\mathcal{A}_{f,\epsilon} \subset \Omega_{\overline{n}^\mu(\epsilon)}$ for every $\epsilon \in (\iota_a + \mu, \epsilon_a - \mu)$ where $\overline{n}^\mu$ is a left continuous decreasing function satisfying $\overline{n}^\mu(\epsilon) \leqslant \overline{n}(\epsilon - \mu)$ for every $\epsilon \in (\iota_a + \mu, \epsilon_a - \mu)$.*

As in the piecewise affine case, this result allow to obtain graphs of upper and lower bounds for the topological entropy of $(\mathcal{K}_{f,\epsilon}, F_{f,\epsilon})$, see Figure 4.

The existence of a partial order as in Proposition 2.2 for general map $f$ (even when restricting the domain of couplings) is unknown. Hence, instead of using order on symbolic sequences, the proof of Theorem 6.2 uses continuity of the map $\psi_{f,\epsilon}(\theta)$ with $f$ uniformly in $\theta$ and strengthening of the admissibility and non-admissibility conditions of the sets $\Omega_n$.

*Proof:* We denote by $\psi_{a,\epsilon}$, the map $\psi_{f_a,\epsilon}$. Uniform continuity of $\psi_{f,\epsilon}$ is given in the next statement.

**Proposition 6.3** *Let $f$ be such that $\|f-f_a\|<\delta$. Then ($a_f \leqslant a$ and) for every $\epsilon \in [0,\epsilon_f)$ and every $\theta\in\Omega$, we have*

$$|\psi_{a,\epsilon}(\theta)-\psi_{f,\epsilon}(\theta)| \leqslant \frac{\delta}{a(1-(a_f(1-2\epsilon))^{-1})}.$$

*Proof of the proposition:* Given a map $f$, let again $f_0$ (resp. $f_1$) be the increasing function obtained by linearly continuating the left (resp. right) monotonic component. The assumption $\|f-f_a\|<\delta$ implies that the differences of inverse maps are bounded as follows

$$\max\left\{\|f_0^{-1}-(f_a)_0^{-1}\|,\|f_1^{-1}-(f_a)_1^{-1}\|\right\}<\delta/a.$$

Consequently, for every $x\in\mathbb{R}^2$, we have

$$\|G^a_{\theta^0}(\mathbf{x})-G^f_{\theta^0}(\mathbf{x})\| \leqslant \delta/a,$$

where $G^a_{\theta^0}$ means the map defined in relation (9) and $G^f_{\theta^0}$ means the corresponding map obtained with $f$ instead of $f_a$. Using also relation (10) for $G^f_{\theta^t}$, we obtain for all $t\in\mathbb{N}$

$$\begin{aligned}
&\|G^a_{\theta^0}\circ\cdots\circ G^a_{\theta^t}(\mathbf{x})-G^f_{\theta^0}\circ\cdots\circ G^f_{\theta^t}(\mathbf{x})\|\\
&\quad\leqslant \|G^a_{\theta^0}\circ G^a_{\theta^1}\circ\cdots\circ G^a_{\theta^t}(\mathbf{x})-G^f_{\theta^0}\circ G^a_{\theta^1}\circ\cdots\circ G^a_{\theta^t}(\mathbf{x})\|\\
&\quad+\|G^f_{\theta^0}\circ G^a_{\theta^1}\circ\cdots\circ G^a_{\theta^t}(\mathbf{x})-G^f_{\theta^0}\circ G^f_{\theta^1}\circ\cdots\circ G^f_{\theta^t}(\mathbf{x})\|\\
&\quad\leqslant \delta/a+b_f^{-1}\|G^a_{\theta^1}\circ\cdots\circ G^a_{\theta^t}(\mathbf{x})-G^f_{\theta^1}\circ\cdots\circ G^f_{\theta^t}(\mathbf{x})\|
\end{aligned}$$

where $b_f=a_f(1-2\epsilon)$. By induction, this inequality shows that, for all $t\in\mathbb{N}$, we have

$$\|G^a_{\theta^0}\circ\cdots\circ G^a_{\theta^t}(\mathbf{x})-G^f_{\theta^0}\circ\cdots\circ G^f_{\theta^t}(\mathbf{x})\| \leqslant \frac{\delta}{a}\sum_{k=0}^{t} b_f^{-k}.$$

The desired conclusion then follows when taking the limit $t\to+\infty$. □

We now present a proof of statements *(i)* and *(ii)* of the Theorem.

For every $n\in\mathbb{N}$, the map $\psi_{a,\epsilon}((00)((10)^n(11))^\infty)$ is a strictly increasing function of $\epsilon$. By definition of $\lambda_{a,n}$ (Lemma 4.3), it results that for every $\eta>0$ (smaller than $1/2-1/a$), there exists $\lambda^\eta_{a,n}\in(0,\lambda_{a,n})$ such that for every $\epsilon\in[0,\lambda^\eta_{a,n})$, we have

$$\psi_{a,\epsilon}((00)((10)^n(11))^\infty)<1/2-\eta.$$

The map $\psi_{a,\epsilon}((00)((10)^n(11))^\infty)$ is a strictly increasing function of $n$ (proof of Lemma 4.3). Thus, $\lambda^\eta_{a,n+1}<\lambda^\eta_{a,n}$. By continuity, we conclude that $\lim\limits_{n\to+\infty}\lambda^\eta_{a,n}=\iota^\eta_a$ where $\iota^\eta_a$ is such that, for every $\epsilon\in[0,\iota^\eta_a)$, we have $\psi_{a,\iota^\eta_a}((00)(10)^\infty)<1/2-\eta$.

In addition, the following limit exist $\lim\limits_{\eta\to0}\lambda^\eta_{a,n}=\lambda_{a,n}$ for every $n$ and $\lim\limits_{\eta\to0}\iota^\eta_a=\iota_a$.

All these limits and the limit $\lim\limits_{n\to+\infty}\lambda_{a,n}=\iota_a$ can be used to show that for every $\mu>0$, there exists $\eta_\mu$ such that

$$\sup_{n\in\mathbb{N}}\lambda_{a,n}-\lambda^{\eta_\mu}_{a,n}<\mu. \tag{11}$$

Given $\mu\in(0,\epsilon_a)$, let $a_\mu$ be defined by $a_\mu(1-2\lambda^{\eta_\mu}_{a,0})=1$ and, given $\alpha\in(a_\mu,a)$, let $\delta>0$ be defined by $\eta_\mu=\frac{\delta}{a(1-(\alpha(1-2\lambda^{\eta_\mu}_{a,0}))^{-1})}$.

By Proposition 6.3, by definition of $\lambda^\eta_{a,n}$ and by Lemma 4.2, it results that, for any map $f$ with $a_f=\alpha$ and $\|f-f_a\|<\delta$ (by relation (11), we have $\epsilon_f>\lambda^{\eta_\mu}_{a,n}>\epsilon_a-\mu$) and for any $\epsilon\in(0,\lambda^{\eta_\mu}_{a,n})$, we have $\psi_{f,\epsilon}(\theta)<1/2$ for all $\theta\in\Omega_n$ such that $\theta^0_0=0$. In short terms, $\Omega_n\in\mathcal{A}_{f,\epsilon}$ for every $\epsilon\in(0,\lambda^{\eta_\mu}_{a,n})$.

The inequality (11) implies that $\iota^{\eta_\mu}_a>\iota_a-\mu$ (we assume that $\mu<\iota_a$). By definition of $\iota^{\eta_\mu}_a$, we conclude that $\psi_{f,\epsilon}(\theta)<1/2$ for every $\theta\in\Omega$ with $\theta^0_0=0$ if $\epsilon\in(0,\iota_a-\mu)$. The first assertion of Statement *(i)* is

proved. The second assertion follows from Statement *(iii)* or can be directly proved by using that $\eta_\mu$ can be chosen so that $\psi_{a,\epsilon}((00)(10)^\infty) > 1/2 + \eta_\mu$ for every $\epsilon > \iota_a + \mu$.

When $\epsilon \in (\iota_a - \mu, \lambda_{a,0}^{\eta_\mu})$, consider the quantity $\underline{n}^\mu(\epsilon) = \sup\{n \in \mathbb{N} \ : \ \epsilon < \lambda_{a,n}^{\eta_\mu}\}$. We have $\Omega_{\underline{n}^\mu(\epsilon)} \subsetneq \mathcal{A}_{f,\epsilon}$ (the strict inequality also comes from the admissibility of $((01)^{\underline{n}^\mu(\epsilon)+1}(10)^{\underline{n}^\mu(\epsilon)+1})^\infty)$ and the inequality (11) implies that $\underline{n}^\mu(\epsilon) \geqslant \underline{n}(\epsilon + \mu)$ for every $\epsilon \in (\iota_a - \mu, \epsilon_a - \mu)$. Statement *(ii)* is proved.

Statement *(iii)* can be proved using the same strategy. That is to say, one has to investigate properties of the smallest numbers $v_{a,n}^\eta \geqslant v_{a,n}$ which are such that, for every $\epsilon \in (v_{a,n}^\eta, \epsilon_a)$ and every $\theta \in \Omega$, either we have $\psi_{a,\epsilon}((00)(10)^{n+1}\theta) < 1/2 - \eta$ or we have $\psi_{a,\epsilon}((10)\theta) > 1/2 - \eta$. The proof is left to the reader. $\square$

# A Proof of properties of the sequence $\{h_n\}$

Firstly, the quantity $h_n$ is equal to the topological entropy of $(\Lambda_n, \sigma)$ because $\Lambda_n$ is the non-wandering set of the system $(\Omega_n, \sigma)$ and $\sigma$ is a continuous map (see e.g. [16]).

Since $(\Lambda_n, \sigma)$ is a subshift of finite type, $h_n$ is the exponential rate of increase of $N_t$, the number of admissible words in $\Lambda_n$ of length $t$, precisely $h_n = \limsup\limits_{t\to+\infty} \frac{\log N_t}{t}$.

To obtain an equation for $h_n$ when $n \geqslant 1$ (the case where $n = 0$ can be computed directly), we compute induction relations for $N_t^0$ (resp. $N_t^{01}$), the number of admissible words in $\Lambda_n$ of length $t$ ending with 00 or with 11 (resp. with 01 but not with $(01)^2$). Let also $N_t^{(01)^k}$ (resp. $N_t^{(10)^k}$) be the number of admissible words of length $t$ ending with $(01)^k$ but not with $(01)^{k+1}$ (resp. with $(10)^k$ but not with $(10)^{k+1}$) where $k \in \{1, \cdots, n\}$.

By symmetry, we have $N_t^{(01)^k} = N_t^{(10)^k}$ for any $k \in \{1, \cdots, n\}$. In addition, it is easy to see that $N_{t+1}^{(01)^k} = N_t^{(01)^{k-1}}$ for all $k \in \{2, \cdots, n\}$. Now, since any word can be followed by 00 or 11, we have

$$N_{t+1}^0 = 2N_t^0 + 4\sum_{k=1}^{n} N_t^{(01)^k} = 2N_t^0 + 4\sum_{k=0}^{n-1} N_{t-k}^{01}$$

Similarly, we successively obtain

$$N_{t+1}^{01} = N_t^0 + \sum_{k=0}^{n} N_t^{(10)^k} = N_t^0 + \sum_{k=0}^{n} N_t^{(01)^k} = N_t^0 + \sum_{k=0}^{n-1} N_{t-k}^{(01)}.$$

These relations show that, if $N_t^0$ and $N_t^{01}$ increase exponentially, then they rates are equal. Assuming that $N_t^0 = c_0\lambda^t$ and $N_t^{01} = c_1\lambda^t$ in these relations forces $\lambda$ to be a solution of $f_n(\lambda) = 0$ where

$$f_n(\lambda) = \lambda^{2(n+1)} - 2\lambda^{2n+1} - (\lambda + 2)\sum_{k=1}^{n} \lambda^{k+n}.$$

In other words, we have $h_n = \log \lambda_n$ where $\lambda_n$ is the largest solution of $f_n(\lambda) = 0$. The properties of the sequence $\{h_n\}$ can now be proved.

Because the alphabet has 4 symbols, each $\lambda_n$ is at most 4. But $f_n(4) > 0$ and then $\lambda_n < 4$ for all $n \geqslant 1$. Moreover $f_n(2) < 0$ and then $\lambda_n > 2$ for every $n \geqslant 1$. The case $n = 0$ can be achieved by direct calculations which show that $f_0(\lambda) = \lambda - 2$ and then $\lambda_0 = 2$.

In addition the definition of $\lambda_n$ implies that $f_{n+1}(\lambda_n) = \lambda_n^{2n+3}(\lambda_n - 4) < 0$ and then $\lambda_{n+1} > \lambda_n$ since $f_{n+1}(\lambda) > 0$ for $\lambda$ sufficiently large.

Finally, we have $\lim\limits_{n\to+\infty} f_n(\lambda) = -\infty$ for every $\lambda \in (2, 4)$ and thus $\lim\limits_{n\to+\infty} \lambda_n = 4$.

# B Proof of Proposition 3.3

We are going to prove the existence of $\eta_a > \iota_a$ such that, for every sequence $\theta \in \Omega$ so that $\theta_0^0 = 0$, either the derivative with respect to $\epsilon$, $\psi'_\epsilon(\theta)$ is positive for every $\epsilon \in (\iota_a, \eta_a)$ or we have $\psi_\epsilon(\theta) < 1/2$ for every

$\epsilon \in (\iota_a, \eta_a)$. By symmetry $\psi_\epsilon(1-\theta) = 1-\psi_\epsilon(\theta)$, it results that every sequence not in $\mathcal{A}_{\epsilon_1}$ is not in $\mathcal{A}_{\epsilon_2}$ for every $\epsilon_2 > \epsilon_1$ and the Proposition follows.

A direct computation shows that $\psi'_\epsilon(\theta)$ is a continuous map of $\theta$ for every $\epsilon$. Using also that $\psi'_{\iota_a}((0\omega)(10)^\infty) > 0$ as follows from relation (6), there exists $t_a \in \mathbb{N}$ such that for every $\Delta \in \Omega$, we have

$$\psi'_{\iota_a}((0\omega)(10)^{t_a}\Delta) \geqslant \frac{\psi'_{\iota_a}((0\omega)(10)^\infty)}{2}.$$

Moreover, it can be shown that the family $\{\psi'_\epsilon(\theta)\}_{\theta\in\Omega}$ is equi-continuous for every $\epsilon$. Together with the previous inequality, we conclude that there exists $\eta_1 > \iota_a$ such that for every $\epsilon \in (\iota_a, \eta_1)$ and every $\Delta \in \Omega$, we have

$$\psi'_\epsilon((0\omega)(10)^{t_a}\Delta) > 0.$$

In addition, by Proposition 2.2-(i), we have $\psi_\epsilon(\theta) \leqslant \max\limits_{1\leqslant i<t_a} \psi_\epsilon((0\omega)(10)^i(11)(10)^\infty)$ for every sequence $\theta = (0\omega)(10)^k\Delta$ with $k < t_a$ and $\Delta^0 \neq 10$. The fact that $\ell_1^{(k)} < 0$ for $k > 0$ and relation (6) then show that

$$\max_{1\leqslant i<t_a} \psi_{\iota_a}((0\omega)(10)^i(11)(10)^\infty) < 1/2.$$

By continuity of the map $\epsilon \mapsto \psi_\epsilon(\theta)$, there exists $\eta_2 > \iota_a$ such that for every $\epsilon \in (\iota_a, \eta_2)$, we have $\max\limits_{1\leqslant i<t_a} \psi_\epsilon((0\omega)(10)^i(11)(10)^\infty) < 1/2$. Letting $\eta_a = \min\{\eta_1, \eta_2\}$ gives the desired statement.

# C Proof of Lemma 4.5

As in the proof of Lemma 4.4, we introduce a mapping which allows to prove that for every $\theta \in P_n$, there exists $\bar\theta \in Q_n$ such that $\psi_\epsilon(\theta) \leqslant \psi_\epsilon(\bar\theta)$.

Given $\theta \in P_n$, consider the first occurrence when the word 11 is not followed by $(10)^n$

$$T(\theta) = \inf\left\{t \geqslant 1 \ : \ \theta^t = 11 \text{ and } \min\left\{p > 0 \ : \ \theta^{t+p} = 11\right\} \leqslant n\right\}.$$

Clearly, we have $Q_n = \{\theta \in P_n \ : \ T(\theta) = +\infty\}$. If $T(\theta) < +\infty$, then let

$$k(\theta) = \min\{k > T(\theta) \ : \ \theta^k = 11\}.$$

Note that the definition of $P_n$ imposes that $T(\theta) > 1$. (Otherwise, the sequence writes $(0\omega)(11)(10)^k(11)\cdots$ for some $k \in \{0, \cdots n-1\}$ and then does not belong to $P_n$.) Similarly, the definition of $T(\theta)$ imposes that $\theta^{k(\theta)+1} = 10$. (Otherwise, the definition of $P_n$ forces $\theta^{k(\theta)}$ to be preceeded by $(10)^n$ which contradicts the definition of $T(\theta)$.)

To define the mapping, we use the following subsets

$$P_\leqslant = \{\theta \in P_n \ : \ T(\theta) \leqslant k_\epsilon\} \quad \text{and} \quad P_> = \{\theta \in P_n \ : \ k_\epsilon < T(\theta) < +\infty\},$$

where $k_\epsilon$ is given in Proposition 2.2-(ii). Let the map $B$ defined in $P_n$ by $B(\theta) = \theta$ if $\theta \in Q_n$, by

$$B(\theta)^t = \begin{cases} \theta^t & \text{if} \quad t \neq T(\theta)-1, T(\theta) \\ 11 & \text{if} \quad t = T(\theta)-1 \\ 10 & \text{if} \quad t = T(\theta) \end{cases}$$

for all $t \in \mathbb{N}$ if $\theta \in P_\leqslant$ and by

$$B(\theta)^t = \begin{cases} \theta^t & \text{if} \quad t \neq k(\theta), k(\theta)+1 \\ 10 & \text{if} \quad t = k(\theta) \\ 11 & \text{if} \quad t = k(\theta)+1 \end{cases}$$

for all $t \in \mathbb{N}$ if $\theta \in P_>$.

The map $B$ is such that $\psi_\epsilon(\theta) \leqslant \psi_\epsilon(B(\theta))$ for every $\theta \in P_n$.

We now detail the study of the behaviour of $T(\theta)$ under iterations. Firstly, note that every sequence $\theta \in P_\leqslant \cup P_>$ writes

$$\theta = (0\omega)(10)^i((11)(10)^n)^j(11)(10)^k(11)(10)^l(11)\cdots,$$

where $i \in \{0, \cdots, n\}$, $j \in \mathbb{N}$, $k \in \{0, \cdots, n-1\}$, $l \in \{0, \cdots, n\}$, $i+k \geqslant n$ if $j = 0$ and $k+l \geqslant n$. In particular, the symbol $\theta^{T(\theta)}$ is the 11 following $((11)(10)^n)^j$, the integer $k = k(\theta) - T(\theta) - 1$ and the inequalities $i+k \geqslant n$ if $j = 0$ and $k + l \geqslant n$ ensure that $\theta \in P_n$.

Assume that $\theta \in P_{\leqslant}$ and consider the case where $j > 0$. We have

$$B(\theta) = (0\omega)(10)^i((11)(10)^n)^{j-1}(11)(10)^{n-1}(11)(10)^{k+1}(11)(10)^l(11)\cdots,$$

and by iterating

$$B^j(\theta) = (0\omega)(10)^i(11)(10)^{n-1}((11)(10)^n)^{j-1}(11)(10)^{k+1}(11)(10)^l(11)\cdots.$$

Now if $i > 0$, then one can apply $B$ once again to obtain

$$B^{j+1}(\theta) = (0\omega)(10)^{i-1}((11)(10)^n)^j(11)(10)^{k+1}(11)(10)^l(11)\cdots.$$

If $i = 0$, then $B^j(\theta) \in O_n \setminus P_n$ so we apply the map $A$ defined in the proof of Lemma 4.4 to obtain (recall that $T$ is only defined for sequences in $P_n$)

$$(A \circ B^j)(\theta) = (0\omega)(10)^n((11)(10)^n)^{j-1}(11)(10)^{k+1}(11)(10)^l(11)\cdots \in P_n$$

In both cases, the resulting value of $T$ at least equals the original one and, if $k + 1 = n$, is larger than the original one. If $k + 1 < n$, one has to apply the same reasoning once again to conclude by iterating that in the case where $\theta \in P_{\leqslant}$ and $j \geqslant 0$, if $i \geqslant n - k$, then

$$B^{(n-k)(j+1)}(\theta) = (0\omega)(10)^{i-(n-k)}((11)(10)^n)^{j+1}(11)(10)^l(11)\cdots,$$

which shows that $T(B^{(n-k)(j+1)}(\theta)) \geqslant T(\theta) + k + 1$. (The case $j = 0$, for which we certainly have $i \geqslant n - k$, can be treated analogously.) If $i < n - k$, then

$$(A \circ (B^{i(j+1)+j}))(\theta) = (0\omega)(10)^n((11)(10)^n)^{j-1}(11)(10)^{k+i+1}(11)(10)^l(11)\cdots,$$

and either $T((A \circ (B^{i(j+1)+j}))(\theta)) \geqslant T(\theta) + k + 1$ if $k + i + 1 = n$ or $(A \circ (B^{i(j+1)+j}))(\theta)$ is a sequence satisfying the assumptions of the previous case where $i \geqslant n - k$.

Assume now that $\theta \in P_{>}$. Then, we have (note that the conditions $k + l \geqslant n$ and $k < n$ impose that $l > 0$)

$$B(\theta) = (0\omega)(10)^i((11)(10)^n)^j(11)(10)^{k+1}(11)(10)^{l-1}(11)(10)^m(11)(10)^p(11)\cdots,$$

where $m, p \in \{0, \cdots, n\}$, $l + m \geqslant n$ and $m + p \geqslant n$. Either $k + 1 = n$ and $T(B(\theta)) \geqslant T(\theta) + 1$ or $k + 1 < n$. In the latter case, if $l - 1 + m \geqslant n$, then one can apply $B$. Otherwise, together with the inequality $l + m \geqslant n$, we have $l + m = n$ and then

$$(A \circ B)(\theta) = (0\omega)(10)^i((11)(10)^n)^j(11)(10)^{k+1}(11)(10)^n(11)(10)^p(11)\cdots,$$

for which we apply the same analysis. Generally one proves that if $\theta \in P_{>}$ and $k + l + m \geqslant 2n$ (i.e. $l - (n - k) + m \geqslant n$) then

$$B^{n-k}(\theta) = (0\omega)(10)^i((11)(10)^n)^{j+1}(11)(10)^{l-(n-k)}(11)(10)^m(11)(10)^p\cdots,$$

and $T(B^{n-k}(\theta)) \geqslant T(\theta) + n + 1$. If $k + l + m < 2n$, then

$$\begin{aligned}&(A \circ B^{l+m-n+1})(\theta) = \\ &\qquad (0\omega)(10)^i((11)(10)^n)^j(11)(10)^{k+l+m-n+1}(11)(10)^n(11)(10)^p(11)\cdots.\end{aligned}$$

If $k+l+m-n+1 = n$, the desired result follows. If $k+l+m-n+1 < n$, then either $k+l+m-n+1+n+p \geqslant 2n$ and the desired result follows by applying $B^{n-(k+n+m-n+1)}$ or $k + l + m + 1 + p < 2n$ and we have the same alternative as before. Since $k + l + m + p + 1 > k + l + m$, by repeating the process, we are sure that the resulting sequence satisfies the inequality $k + l + m \geqslant 2n$ after a finite number of iterations.

We have shown that, for every sequence $\theta \in P_{\leqslant} \cup P_{>}$, there exists an integer $k$ such that

$$T((A \circ B)^k(\theta)) > T(\theta) \quad \text{and} \quad \psi_\epsilon(\theta) \leqslant \psi_\epsilon((A \circ B)^k(\theta)).$$

As in the proof of Lemma 4.4, we conclude that the limit $\lim_{k \to +\infty} (A \circ B)^k(\theta)$ exists, belongs to $Q_n$ and has a larger value of $\psi_\epsilon$.

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